Definition and analysis of a two dimensional diffusion process with boundary and jumps

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DEFINITION AND ANALYSIS OF A TWO DIMENSIONAL DIFFUSION PROCESS WITH BOUNDARY AND JUMPS

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Abstract

We analyze a two dimensional diffusion process in the positive quarter plane with absorptions on the boundaries and jumps.
In the first section, the sample paths of the process are constructed using stochastic differential equations. In the second section, we determine the invariant measure in the case of constant coefficients by solving a boundary value problem on an hyperbola. Thus, a generalization of the one dimensional case (see for instance [GUI,SKO] -or [GEL,MIT] for queueing applications-) is obtained.

Résumé

Nous analysons un processus de diffusion bi-dimensionnel dans le quart de plan comportant des absorptions sur les frontières et des sauts.
Dans la première section, nous construisons les trajectoires de ce processus au moyen d'équations différentielles stochastiques. Dans la seconde, nous calculons la mesure invariante dans le cas où les coefficients sont constants, comme solution d'un problème frontière sur une hyperbole. Ce travail prolonge l'étude du cas unidimensionnel (voir par exemple [GUI,SKO]- ou [GEL,MIT] pour des applications aux files d'attente -).

SECTION 1

I. DEFINITION OF THE STOCHASTIC PROCESS

I.1. Introduction

We define here a discontinuous stochastic process $(\xi(t))_{t \geq 0}$ with state space $\mathbb{R}^+ \times \mathbb{R}^+$ whose sample paths are recursively given by a set of stochastic differential equations (S.D.E) and a set of 2 independent marked Poisson processes determining certain sojourn times and jump magnitudes. Let us denote by :

- D the positive quarter plane except the axes : $D = \mathbb{R}^{+*} \times \mathbb{R}^{+*}$

- ∂D the boundary : $\partial D = \{0\} \times \mathbb{R}^{+*} \cup \mathbb{R}^{+*} \times \{0\}$

- I (resp. II) the axis $\mathbb{R}^{+*} \times \{0\}$ (resp. $\{0\} \times \mathbb{R}^{+*}$)

The sample path of $\xi(t)$ starts at time t=0, say somewhere in D and is the solution of a two-dimensional stochastic differential equation, until it hits the boundary ∂D. (If it does never reach ∂D, the path is hence defined over $[0,\infty)$). When this boundary is reached, say at point $\{0\} \times \{y\}$, $y > 0$, the path is "absorbed" by the boundary "II" for a period which will not exceed the next point of the first Poisson process after the absorption time. During this period, the path is given as the solution of a one-dimensional stochastic differential equation until it hits the point $\{0\} \times \{0\}$. Assume it reaches " 0" before the next point of the first Poisson process, then the path is absorbed by "0" where it remains up to the next point of the second Poisson process after this last absorption time. When this next point arrives, the path jumps instantaneously to a new location determined by the mark of this point. This mark belongs to $\mathbb{R}^+ \times \mathbb{R}^+$ and is either of the type $(x,0)$, $x > 0$ or $(0,y)$, $y > 0$. From this new location which is either in $\mathbb{R}^{+*} \times \{0\}$ or in $\{0\} \times \mathbb{R}^{+*}$, is initialized a new one-dimensional period which will be a stochastic replica of the previous one (but the initial position). Assume on the contrary that next point of the first Poisson process is before the hiting time of $\{0\}$ by the one-dimensional diffusion, then, at that point, the path jumps instaneously, the jump being orthogonal to the $\{0\} \times \mathbb{R}^+$ boundary and with a magnitude $z > 0$ given by the mark of the point. This initializes a new bidimensional "period" of the process which will be a stochastic replica of the first one (but the new initial conditions).

I.2. Notations and assumptions

Let $(W_1(t))_{t \geq 0}$ and $(W_2(t))_{t \geq 0}$ be two independent brownian motions will be the basic brownian motions of the S.D.E. Moreover, let us define two processes, M_t and N_t which are independent and also independent of the two brownian motions. $(M_t)_{t \geq 0}$ (resp. $(N_t)_{t \geq 0}$) is a right continuous jump process with state space $\mathbb{R}^+ \times \mathbb{R}^+$. The jump times form a Poisson process with intensity λ (resp. μ). The process M starts from (0,0) and jumps at each date of the Poisson process, the magnitude of the successive jumps being independent of the Poisson process. Denoting as (U_n, V_n) (resp. (X_n, Y_n) the n-th mark of $(M(t))_{t \geq 0}$, (resp. $(N(t))_{t \geq 0}$) we assume for them the following distribution functions :

$$(1.1) \quad \begin{cases} P[U_n = 0,\ V_n \leq y] = P_2 H_2(y),\ P[U_n \leq y,\ V_n = 0] = P_1 H_1(y) \\ \text{where } P_1 + P_2 = 1,\ H_i(0)=0,\ H_i(\infty)=1,\ \int_0^{\infty} x^2\, dH_i(x) < \infty \end{cases}$$

$$(1.2) \quad \begin{cases} P[X_n = 0;\ Y_n \leq y] = R_2 K_2(y),\ P[X_n \leq y,\ Y_n \leq 0] = R_1 K_1(y) \\ \text{where } R_1 + R_2 = 1,\ K_i(0)=0,\ K_i(\infty)=1,\ \int_0^{\infty} x^2 dK_i(x) < \infty \end{cases}$$

The remaining definitions concern the one and two-dimensional S.D.E. The assumptions make sure the existence and uniqueness of the S.D.E solutions in their definition domains (see [Fried])

Let $b_i(x)$ and $\sigma_i(x)$, i=1,2 be real valued functions which are measurable in $(x) \in \mathbb{R}^+$ - $B(x) = (B_1(x), B_2(x))$, (resp. $\Sigma(x) \doteq \begin{pmatrix} \Sigma_{11}(x) & \Sigma_{12}(x) \\ \Sigma_{21}(x) & \Sigma_{22}(x) \end{pmatrix}$) be a two-dimensional real valued vector (resp. matrix) which is measurable in $x \in \mathbb{R}^+ \times \mathbb{R}^+$ satisfying $\forall\ R \in \mathbb{R}^+$, $\forall\ x, x' \in \mathbb{R}^+$, $|x|$, $|x'| \leq R$, $\exists\ K_R$, K and $\sigma_i > 0$ such that :

$$(1.3)\quad \begin{cases} |b_i(x) - b_i(x')| \leq K_R|x-x'| \\ |\sigma_i(x) - \sigma_i(x')| \leq K_R|x-x'| \\ |b_i(x)| \leq K \\ \sigma_i \leq |\sigma_i(x)| \leq K \end{cases}$$

$\forall R \in \mathbb{R}^+$, $\forall x, x' \in \mathbb{R}^+ \times \mathbb{R}^+ \;/\; |x|, |x'| \leq R$, H_R, H such that :

$$(1.4)\quad \begin{cases} |B(x) - B(x')| \leq H_R|x-x'| \\ |\Sigma(x) - \Sigma(x')| \leq H_R|x-x'| \\ |B(x)| \leq H \\ |\Sigma(x)| \leq H \end{cases}$$

(we denote as $|\ |$ the classical vector or matrix norm in $\mathbb{R}^n$).

It is furthermore assumed that the quadratic form associated to A(x) $A(x) \overset{\text{def}}{=} \Sigma(x).\Sigma(x)^t$ (where Σ^t is the transpose of Σ) is uniformly elliptic i.e, $\exists\ \mu > 0$ such that if $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, $\forall x,y \in \mathbb{R}^+ \times \mathbb{R}^+$,

$$(1.5)\qquad a(x,y)x^2 + 2b(x,y)\ x.y + c(x,y)y^2 \geq \mu(x^2+y^2).$$

We also assume that the biggest eigenvalue of A(x) for $x \in \mathbb{R}^+ \times \mathbb{R}^+$ is bounded by $\rho > 0$. We shall denote by L_X (resp. L_Y) the following one-dimensional partial differential operator :

$$(1.6)\quad \begin{cases} L_X = \frac{1}{2}\, a_1(x)\, \frac{d^2}{dx^2} + d_1(x)\, \frac{d}{dx} & \text{where } \begin{cases} a_1(x) = \sigma_1^2(x) \\ d_1(x) = b_1(x) \end{cases} \\ \text{resp. } L_Y = \frac{1}{2}\, a_2(x)\, \frac{d^2}{dx^2} + d_2(x)\, \frac{d}{dx} & \text{where } \begin{cases} a_2(x) = \sigma_2^2(x) \\ d_2(x) = b_2(x) \end{cases} \end{cases}$$

We shall denote by L the following two-dimensional partial differential operator :

$$(1.7)\quad \begin{cases} L = \frac{1}{2}[a(x,y)\, \frac{\partial^2}{\partial x^2} + 2b(x,y)\, \frac{\partial}{\partial x}\frac{\partial}{\partial y} + c(x,y)\, \frac{\partial^2}{\partial y^2}] + d(x,y)\, \frac{\partial}{\partial x} + e(x,y)\, \frac{\partial}{\partial y} \\ \text{where } d(x,y) = B_1(x,y), \quad e(x,y) = B_2(x,y) \end{cases}$$

I.3. Definition of the process

We shall denote as (Ω,F,P) the basic probability space. Ω will be taken as the product of the tow-dimensional brownian motion canonical space and the canonical spaces of the 2 supplementary marked Poisson processes. We shall denote as $(F_t)_{t \geq 0}$ the filtration $\sigma(W(s),M(s),N(s),s \leq t)$ where $W(t) = (W_1(t),W_2(t))$ and use the following results :

Lemma 1.

Let Γ be any finite F_t stopping time, then, $(W(t+\Gamma)-W(\Gamma),M(t+\Gamma)-M(\Gamma),$ $N(t+\Gamma)-N(\Gamma))_{t \geq 0}$ is a stochastic process independent of $F_\Gamma = \sigma\{B \in F/B \cap \{\Gamma \leq t\} \in F_t \quad \forall\, t \geq 0\}$ and with the same law as $(W(t),M(t),N(t))_{t \geq 0}$ $\square$

Proof of Lemma 1.

Let Γ be a finite F_t stopping time and let

$$\Gamma_n = \sum_{k \geq 0} k2^{-n}\, 1_{[(k-1)2^{-n} < \Gamma \leq k2^{-n}]}$$

Let $d \in \mathbb{N}$, $B \in F_\Gamma$ and f be a continuous bounded function from $\mathbb{R}^{4d} \to \mathbb{R}$. Let $0 < t_1,\ldots < t_d$ and put for any fixed $i \in \{1,2\}$, x_o, x_1, $x_2 > 0$:

$$\begin{aligned} e(t) = f\,[&W_1(t_1+t) - W_1(t), \ldots, W_1(t_d+t) - W_1(t),\\ &W_2(t_1+t) - W_2(t), \ldots, W_2(t_d+t) - W_2(t),\\ &M_o(t_1+t) - M(t), \ldots, M(t_d+t) - M(t),\\ &N\ (t_1+t) - N(t), \ldots, N(t_d+t) - N(t)]. \end{aligned}$$

Since $\Gamma_n \downarrow \Gamma$, when $n \uparrow \infty$, $e(\Gamma_n)$ tends to $e(\Gamma)$ as $n \uparrow \infty$. Furthermore :

$$B \cap (\Gamma_n = k2^{-n}) \in F_{k2^{-n}}$$

Hence

$$E[1_B.e(\Gamma)] = \lim_{n\uparrow\infty} E[1_B.e(\Gamma_n)]$$
$$= \lim_{n\uparrow\infty} \sum_{k=0}^{\infty} E[1_{B \cap (\Gamma_n = k2^{-n})}.e(k2^{-n})]$$

Clearly $W_i(t_j+t) - W_i(t)$ is independent of F_t and so are $((M(t_j+t)-M(t)),(N(t_j+t)-N(t)))$ for any $t > 0$ and $t_j > 0$. Using this property for $t = k2^{-n}$, we get :

$$E[1_B.e(\Gamma)] = \lim_{n\to\infty} \sum_{k=0}^{\infty} E[B \cap (\Gamma_n = k2^{-n})].E[e(k2^{-n})]$$

Using now the properties of Brownian motion and Poisson processes $E[1_B.e(\Gamma)] = E[e(O)]E[B]$, completing the proof. □

The remaining part of this section is devoted to the construction of a F_t adapted, right continuous stochastic process $(\xi(t))_{t\geq 0}$ on (Ω,F,P) with state space $\mathbb{R}^+ \times \mathbb{R}^+$.

We shall denote as (e_1,e_2) the orthonormal base of the plane and use the shift operators $(\Theta_t)_{t\geq 0}$.

Consider the unique solution (uniqueness and existence are due to our assumptions on Σ and B) of the following bidimensional S.D.E :

$$\begin{cases} d\,\eta(s) = \Sigma(\eta(s))\,dW(s) + B(\eta(s)ds \\ \eta(0) = x \in \mathbb{R}^+ \times \mathbb{R}^+,\ 0 \leq s \leq \tau \wedge t = \inf\{s \geq 0 \,/\, \eta(s) \notin D\} \wedge t \end{cases}$$

We define our process ξ when originated from x between time O up to $(\tau \wedge t)$ as being equal to this unique solution during this time interval. Hence we have :

$$(1.8) \qquad \xi(t\wedge\tau) = x + \int_0^{\tau\wedge t} \Sigma(\xi(s))dW(s) + \int_0^{\tau\wedge t} B(\xi(s))ds$$

We define now a "free continuation" from time $\tau \wedge t$ up to $(s + \tau \wedge t) \wedge t$ when M is frozen. It originates from $\xi(\tau \wedge t)$. Roughly speaking it consists of successively diffusing on either "I" or "II" then being absorbed and remain in "O" then jump into either "I" or "II" and so forth. Consider the following one-dimensional S.D.E.'s, i=1,2

$$\begin{cases} d\,\eta_i(u) = \sigma_i(\eta_i(u))\,dW_i(u) + b_i(\eta_i(u))du, \quad \tau \wedge t \le u \le \gamma_i \circ \theta_{\tau\wedge t} \wedge t \\ \text{with initial condition : } \quad \eta_i(\tau \wedge t) = \xi_i(\tau \wedge t) \\ \text{where } \gamma_i \circ \theta_{\tau \wedge t} = \inf\{s \ge \tau \wedge t \ / \ \eta_i(s) = 0\} \wedge t \text{ (notice that at least} \\ \qquad \text{one of these stopping times is equal to } \tau \wedge t). \end{cases}$$

The initial condition for this S.D.E's are square integrable and independent of the future of W after $\tau \wedge t$, as a consequence of lemma 1. Hence each of these differential equations has a unique solution $\eta_i(u)$, $\tau \wedge t \le u \le \gamma_i \circ \theta_{\tau\wedge t} \wedge t$. We define our two dimensional free continuation $\hat{\xi} \overset{\text{def}}{=} (\hat{\xi}_1, \hat{\xi}_2)$ between $\tau \wedge t$ and $(\Gamma \circ \theta_{\tau \wedge t} \wedge t)$ where $\Gamma \circ \theta_{\tau \wedge t} = (\gamma_1 \circ \theta_{\tau \wedge t}) \vee (\gamma_2 \circ \theta_{\tau \wedge t})$ as follows :

$$(1.9) \qquad \begin{cases} \hat{\xi}_i(u) = \eta_i(u), \quad \tau \wedge t \le u \le \gamma_i \circ \theta_{\tau\wedge t} \wedge t \\ \hat{\xi}_i(u) = \eta_i(\gamma_i \circ \theta_{\tau \wedge t} \wedge t), \; \gamma_i \circ \theta_{\tau\wedge t} \wedge t \le u \le \Gamma \circ \theta_{\tau \wedge t} \wedge t \end{cases}$$

We write hence :

$$(1.10) \qquad \begin{cases} \hat{\xi}(\Gamma \circ \theta_{\tau\wedge t} \wedge t) = \xi(\tau \wedge t) + \sum_{i=1}^{2} \\ \left\{ \int_{\tau \wedge t}^{\gamma_i \circ \theta_{\tau \wedge t} \wedge t} \sigma_i(\hat{\xi}_i(u))dW_i(u) + b_i(\hat{\xi}_i(u))du \right\} e_i \end{cases}$$

Let T be any finite F_t stopping time. Let $\delta \circ \theta_T$ be the following F_t stopping time :

$$\delta \circ \theta_T = \inf\{u \ge T \ / \ \Delta N(u) \ne 0\} \wedge t \qquad (\text{where } \Delta(u) = N(u) - N(u^-))$$

We go on defining $\hat{\xi}$ after $\Gamma \circ \theta_{\tau \wedge t} \wedge t$ as follows :

$$(1.11)\qquad \begin{cases} \hat{\xi}(u) = \hat{\xi}(\Gamma \circ \theta_{\tau \wedge t} \wedge t),\ \Gamma \circ \theta_{\tau \wedge t} \wedge t \le u < \delta \circ \theta_{(\Gamma \circ \theta_{\tau \wedge t}) \wedge t} \\ \hat{\xi}(\delta \circ \theta_{(\Gamma \circ \theta_{\tau \wedge t}) \wedge t} = \Delta N(\delta \circ \theta_{(\Gamma \circ \theta_{\tau \wedge t}) \wedge t} \end{cases}$$

Then we iterate the definition procedure of the two dimensional S.D.E taking now the $\xi_i(\delta \circ \theta_{\Gamma \circ \theta_{\tau \wedge t}} \wedge t)$'s as the new initial conditions. This is legal from our assumptions in equation (1.2) and from lemma 1 . We obtain hence the following definition for $\hat{\xi}((\tau \wedge t + s) \wedge t)$ (by iterating indefinetly this definition procedure):

$$(1.12)\qquad \begin{cases} \hat{\xi}((\tau \wedge t + s) \wedge t) = \xi(\tau \wedge t) + \\ + \left\{ \int_{\tau \wedge t}^{\gamma_1 \circ \theta_{\tau \wedge t}) \wedge (s + \tau \wedge t) \wedge t} \sigma_1(\hat{\xi}_1(u))dW_1(u) + b_1(\hat{\xi}_1(u))du \right\} e_1 \\ + \left\{ \int_{\tau \wedge t}^{(\gamma_2 \circ \theta_{\tau \wedge t})\ (s + \tau \wedge t) \wedge t} \sigma_2(\hat{\xi}_2(u))dW_2(u) + b_2(\hat{\xi}_2(u))du \right\} e_2 \\ + \Delta N\ (\delta \circ \theta_{(\Gamma \circ \theta_{\tau \wedge t}) \wedge (s + \tau \wedge t) \wedge t} \wedge (s + \tau \wedge t) \wedge t)) \\ + \left\{ \int_{\delta \circ \theta_{\Gamma \ldots} \wedge (s + \tau \wedge t) \wedge t}^{(\gamma_1 \circ \theta_{\delta \circ \theta_{\Gamma \ldots}}) \wedge (s + \tau \wedge t) \wedge t} \sigma_1(\hat{\xi}_1(u))dW_1(u) + G_1(\hat{\xi}_1(u))du \right\} e_1 \\ + \left\{ \int_{\delta \circ \theta_{\Gamma \ldots} \wedge (s + \tau \wedge t) \wedge t}^{(\gamma_2 \circ \theta_{\delta \circ \theta_{\Gamma \ldots}})(s + \tau \wedge t) \wedge t} \sigma_2(\hat{\xi}_2(u))dW_2(u) + G_2(\hat{\xi}_2(u))du \right\} e_2 \\ + \Delta N\ (\delta \circ \theta_{\Gamma \circ \theta_{(\delta \circ \theta_{\Gamma \ldots})}} \wedge (s + \tau \wedge t) \wedge t \wedge (s + \tau \wedge t)) \\ + \left\{ \int_{\delta \circ \theta_{\Gamma \circ \theta_{(\delta \circ \theta_{\Gamma \ldots})}} \ldots}^{\gamma_1 \ldots} \ldots \right\} e_1 + \ldots + \ldots \end{cases}$$

Remark that the number of non zero terms in this definition is a.s. finite : Each sojourn time in "0" has an exponentially distributed duration and is independent of the other sojourn times. Hence we a.s have a finite number of δ's smaller than $(\tau \wedge t + s) \wedge t$. Notice also it is possible to replace s, in this definition, by any finite F_t stopping time. Consider now the following F_t stopping time :

$$(1.13)\quad \begin{cases} \eta \circ \theta_{\tau \wedge t} = \\ 1_{[\tau \geq t]} \cdot t + 1_{[\tau < t]} \inf\{s \geq \tau \;/\; (\hat{\xi}(s) \in \text{"I"} \\ \text{and } \Delta M(s).e_2 > 0) \text{ or } (\hat{\xi}(s) \in \text{"II"} \text{ and } \Delta M(s).e_1 > 0)\} \end{cases}$$

We go on defining ξ between $\tau \wedge t$ and $\eta \circ \theta_{\tau \wedge t}$ as being equal to the free continuation between these stopping times, more precisely

$$\begin{cases} \xi(u) = \hat{\xi}(u) & \tau \wedge t \leq u < \eta \circ \theta_{\tau \wedge t} \\ \xi(\eta \circ \theta_{\tau \wedge t}) = \hat{\xi}(\eta \circ \theta_{\tau \wedge t}) + \Delta M(\eta \circ \theta_{\tau \wedge t}) \end{cases}$$

The random variable $\hat{\xi}(\eta \circ \theta_{\tau \wedge t}^{-})$ is quare integrable as well as $\Delta M(\eta \circ \theta_{\tau \wedge t})$, (see equation (1.1)). From this plus lemma 1, it makes sens to take $\xi(\eta \circ \theta_{\tau \wedge t})$ as the initial condition for a new two-dimensional period. We can iterate this definition procedure. Denote as $\tau_i \wedge t$ $i \geq 1$, the minimum of t and of the i-th successive hiting time of ∂D. Similarly, let $\eta_{i \wedge t} \overset{\text{def}}{=} \eta \circ \theta_{\tau_i \wedge t}$. Denote as $\Gamma_j^i \wedge \eta_i \wedge t$ the minimum of $\eta_i \wedge t$ and the j-th, $j \geq 1$, hiting time of " 0" after $_i$. Let also $\delta_j^i \wedge \eta_i \wedge t \overset{\text{def}}{=} \delta \circ \theta_{\Gamma_j^i \wedge \eta_i \wedge t}$. We define $\xi(t)$ as :

$$(1.14)\quad \begin{cases} \xi(t) = x + \displaystyle\int_0^{\tau_1 \wedge t} \Sigma(\xi(s))dW(s) + B(\xi(s))ds \\ + \displaystyle\sum_{i=1}^{\infty} \left\{ \int_{\tau_i \wedge t}^{\Gamma_1^i \wedge \eta_i \wedge t} \sigma(\xi(s))dW(s) + b(\xi(s))ds \right. \\ \displaystyle\sum_{j=1}^{\infty} [\Delta N(\delta_j^i \wedge \eta_i \wedge t) + \int_{\delta_j^i \wedge \eta_i \wedge t}^{\Gamma_{j+1}^i \wedge \eta_i \wedge t} \sigma(\xi(s))dW(s) + b(\xi(s))ds] \\ + \Delta M(\eta_i \wedge t) \\ \left. + \displaystyle\int_{\eta_i \wedge t}^{\tau_{i+1} \wedge t} \Sigma(\xi(s))dW(s) + B(\xi(s))ds \right\} \end{cases}$$

where the vectorial integrals "$\int \sigma \, dw(s) + b \, ds$ denote sums of two one-dimensional integrals as before. Notice that the R.H.S of (1.14) is well defined since the number of non zero terms in this summation is a.s finite : between two successive τ_i's ellapses at least an independant exponentially distributed delay. Hence there exists a.s a finite i such that τ_i is greater than t, as well as all of the stopping times from τ_i.

II. MARKOVIAN ANALYSIS

II.1. The Markov property

Theorem 2 $(\xi(t))_{t \geq 0}$ is Markov Process. □

Proof : Consider $(\xi(y))\quad 0 \leq y \leq t$ as defined in the preceding section For u / $0 < u < t$, we derive the following decomposition : of the R.H.S of (1.14) :

$$
\begin{aligned}
(2.1)\qquad \text{R.H.S.} \overset{a.s}{=} \; & x + \int_0^{\tau_1 \wedge u} \underline{\Sigma(\xi(s))dW(s) + B(\xi(s))ds} \; + \\
& + \int_{\tau_1 \wedge u}^{\tau_1 \wedge t} \Sigma(\xi(s))dW(s) + B(\xi(s))ds \\
& + \sum_{i=1}^{\infty} \Bigg\{ \Bigg(\int_{\tau_i \wedge t}^{(\Gamma_1^i \wedge \eta_i \wedge (u \vee (\tau_i \wedge t))} \underline{\sigma(\xi(s))dW(s)+b(\xi(s))ds} + \int_{(\Gamma_1^i \wedge \eta_i \wedge (u \vee (\tau_1 \wedge t))}^{\Gamma_1^i \wedge \eta_i \wedge t} \sigma(\xi(s))dW(s)+b(\xi(s))ds \Bigg) \\
& + \sum_{j=1}^{\infty} \Bigg[\underline{\Delta N(\delta_i^j \wedge \eta_i \wedge u)} + \Delta N((\delta_i^j \wedge \eta_i \wedge t) \vee u) \\
& \int_{\delta_j^i \wedge \eta_i \wedge t}^{\Gamma_{j+1}^i \wedge \eta_i \wedge (u \vee (\delta_j^i \wedge \eta_i \wedge t)} \underline{\sigma(\xi(s))dW(s)+b(\xi(s))ds} + \int_{\Gamma_{j+1}^i \wedge \eta_i \wedge (u \vee (\delta_j^i \wedge \eta_i \wedge t))}^{\Gamma_{j+1}^i \wedge \eta_i \wedge t} \sigma(\xi(s))dW(s)+b(\xi(s))ds \Bigg] \\
& + \underline{\Delta M(\eta_i \wedge u)} + \Delta M(u \vee (\eta_i \wedge t)) \\
& + \int_{\eta_i \wedge t}^{\tau_{i+1} \wedge (u \vee (\eta_i \wedge t))} \underline{\Sigma(\xi(s))dW(s)+B(\xi(s))ds} \\
& + \int_{\tau_{i+1} \wedge (u \vee (\eta_i \wedge t))}^{\tau_{i+1} \wedge t} \Sigma(\xi(s))dW(s)+B(\xi(s))ds \Bigg\}
\end{aligned}
$$

For the underlined stochastic integrals in (2.1) hold equalities like the following :

$$\int_{\tau_i \wedge t}^{\Gamma_1^i \wedge \eta_i \wedge (u \vee (\tau_i \wedge t))} \sigma(\xi(s))dW(s)+b(\xi(s))ds$$

$$= \int_{\tau_i \wedge u}^{\Gamma_1^i \wedge \eta_i \wedge u} \sigma(\xi(s))dW(s) + b(\xi(s))ds$$

(see I p)

which is obtained from the definition of stochastic integrals with stopping times and from the identity :

$$\{s \;/\; [\Gamma_1^i \wedge \eta_i \wedge (u \vee (\tau_i \wedge t))] \le s \le [\tau_i \wedge t]\} =$$

$$\{s \;/\; [\Gamma_1^i \wedge \eta_i \wedge u] \le s \le [\tau_i \wedge u]\}$$

Collecting all together the terms which are underlined in (2.1) and using the above identities, we obtain an expression which coincides with the definition of $\xi(u)$. Using similar transformations for the remaining terms, we get :

$$\xi(t) = \xi(u) + \int_u^{u \vee (\tau_1 \wedge t)} \Sigma(\xi(s))dW(s)+B(\xi(s))ds$$

$$+ \sum_{i=1}^{\infty} \left\{ \int_{u \vee (\tau_1 \wedge t)}^{u \vee (\Gamma_1^i \wedge \eta_i \wedge t)} \sigma(\xi(s))dW(s)+b(\xi(s))ds \right.$$

$$+ \sum_{j=1}^{\infty} \left[\Delta N((\delta_i^j \wedge \eta_i \wedge t) \vee u) + \int_{u \vee (\delta_j^i \wedge \eta_i \wedge t)}^{u \vee (\Gamma_{j+1}^i \wedge \eta_i \wedge t)} \sigma(\xi(s))dW(s)+b(\xi(s))ds \right]$$

$$+ \Delta M((\eta_i \wedge t) \vee u) + \int_{u \vee (\tau_i \wedge t)}^{u \vee (\tau_{i+1} \wedge t)} \Sigma(\xi(s))dW(s)+B(\xi(s))ds$$

Keeping in mind the obvious identity :

$\int_\alpha^{\alpha \vee \beta} = 0$ if $\beta < \alpha$, we introduce

$\tau_1 \circ \theta_u = \inf \{s \geq u \mid \xi(s) \notin D\}$ (plus similar definitions for $\tau_i \circ \theta_u$, $\Gamma_j^i \circ \theta_u$, $\delta_j^i \circ \theta_u$, $\eta_i \circ \theta_u$). This allows us to rewrite the last expression as :

$$\xi(t) = \xi(u) + \int_u^{(\tau_1 \circ \theta_u) \wedge t} \Sigma(\xi(s))dW(s)+B(\xi(s))ds \tag{2.2}$$
$$+ \sum_{i=1}^{\infty} \Bigg\{ \int_{(\tau_1 \circ \theta_u) \wedge t}^{(\Gamma_1^i \circ \theta_u) \wedge (\eta_i \circ \theta_u) \wedge t} \sigma(\xi(s))dW(s)+b(\xi(s))ds$$
$$+ \sum_{j=1}^{\infty} \Bigg[\Delta N((\delta_j^i \circ \theta_u) \wedge (\eta_i \circ \theta_u) \wedge t)$$
$$+ \int_{(\delta_j^i \circ \theta_u) \wedge (\eta_i \circ \theta_u) \wedge t}^{(\Gamma_{j+1}^i \circ \theta_u) \wedge \eta_i \wedge t} \sigma(\xi(s))dW(s)+b(\xi(s))ds \Bigg]$$
$$+ \Delta M((\eta_i \circ \theta_u) \wedge t)$$
$$+ \int_{(\eta_i \circ \theta_u) \wedge t}^{(\tau_{i+1} \circ \theta_u) \wedge t} \Sigma(\xi(s))+B(\xi(s))ds \Bigg\}$$

proving the Markov property.

II.2. The Infinitesimal generator

Theorem 3. Let $\mathcal{A}$ be the infinitesimal generator of $(\xi(s))$. Let f be a twice continuously differenciable function on $\mathbb{R}^+ \times \mathbb{R}^+$ which is assumed to be bounded, and with bounded derivatives, then $f \in \mathcal{D}(\mathcal{A})$, the domain of $\mathcal{A}$ and we have :

$$\mathcal{A} f(x,y) = L\, f(x,y) \qquad x > 0, \quad y > 0 \tag{2.3}$$

$$\begin{cases} \mathcal{A} f(x,0) = L_X f(x,0) + g_2 \int_0^\infty (f(x,y)-f(x,0))dH_2(y) \\ \mathcal{A} f(0,y) = L_Y f(0,y) + g_1 \int_0^\infty (f(x,y)-f(0,y))dH_1(x) \end{cases} \tag{2.4}$$

$$\mathcal{A} f(0,0) = q_1 \int_0^\infty (f(x,0)-f(0,0))dK_1(x) + q_2 \int_0^\infty (f(0,y)-f(0,0))dK_2(y) \qquad \square \tag{2.5}$$

where $g_1 = \lambda P_1$, $g_2 = \lambda P_2$

$q_1 = \mu R_1$, $q_2 = \mu R_2$

Proof Using the notations introduced previously, we have to verify the existence and the value of :

$$f(x,y) = \lim_{h \downarrow 0} \frac{E[f(\xi(h))-f(x,y)\,|\,\xi(o) = (x,y)]}{h}$$

We consider first the case $x > 0$, $y > 0$.

$$E[f(\xi(h))-f(x,y)\,|\,\xi(0) = (x,y)] =$$

$$E[f(\xi(h \wedge \tau_1))-f(x,y)+f(\xi(h))-f(\xi(\tau_1 \wedge h)\,|\,\xi(o)=(x,y)]$$

Using Ito's formula, we get

$$E[f(\xi(h \wedge \tau_1))-f(x,y)\,|\,\xi(o)=(x,y)] = E[\int_o^{\tau_1 \wedge h} Lf(\xi(s))ds\,|\,\xi(o)=(x,y)]$$

so that

$$\exists \lim_{h \to 0} \frac{1}{h} E[f(\xi(h \wedge \tau_1))-f(x,y)\,|\,\xi(o)=(x,y)] = Lf(x,y)$$

Furthermore

$$|E[\frac{1}{h} f(\xi(h))-f(\xi(h \wedge \tau_1)\,|\,\xi(o)=(x,y)]| \le \frac{2Q}{h} p[\tau_1 \le h\,|\,\xi(o)=(x,y)]$$

where $Q = \max_{x \in D} |f(x)|$

It holds (see [McKean] p. 93)

$$P\,[\max_{s \le h} |\xi(s)-\xi(o)| \ge R\,] \le \exp[-\frac{R^2}{4\rho h}]$$

where ρ is the biggest eigen value of the positive definite matrix

$$A(x) = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad \text{for } x \in \mathbb{R}^+ \times \mathbb{R}^+.$$

Taking $0 < R < x \wedge y$ we get :

$$P\,[(\tau_1 \leq h)\,|\,\xi(o) = (x,y)] \leq P[\max_{s\leq h}|\xi(s)-\xi(o)| \geq R\,|\,\xi(o)=(x,y)] \leq \exp[-\frac{R^2}{4\rho h}]$$

so that :

$$\lim_{h\to 0} \frac{1}{h} P[(\tau \leq h)\,|\,\xi(o)=(x,y)] = 0$$

completing the proof of (2.3).

We consider now the case $x > 0$, $y = 0$
The solution of the SDE

$$\begin{cases} d\xi_1(t) = \sigma_1(\xi_1(t))dW_1(t) + b_1(\xi_1(t))dt \\ t \geq 0, \quad \xi_1(0) = x \end{cases}$$

satisfies (see again [3] p. 93) :

$$P[\max_{s\leq h} |\xi_1(s)-\xi_1(o)| \geq R] \leq \exp\,[-\frac{R^2}{2\sigma_1^2 h}]$$

Hence, if we denote as $\gamma_1 = \inf\,\{t \geq 0/\xi_1(t) = 0\}$, we get :

(2.6) $$\lim_{h\searrow 0} \frac{1}{h} P\,[\gamma_1 \leq h\,|\,\xi_1(o)=x] = 0$$

(2.7) $$\begin{aligned} &E[f(\xi(h))-f(x,0)\,|\,\xi(o)=(x,o)] = \\ &E[f(\xi(\gamma_1 \wedge \eta \wedge h))-f(x,o)\,|\,\xi(o)=(x,o)] + \\ &E[f(\xi(h))-f(\xi(\gamma_1 \wedge \eta \wedge h)\,|\,\xi(o)=(x,o)] \end{aligned}$$

We have :

$$\begin{aligned} &\frac{1}{h}|E[f(\xi(h)-f(\xi(\gamma_1 \wedge \eta \wedge h)\,|\,\xi(o)=(x,o)]| \leq \\ &\frac{2Q}{h} P[\gamma_1 \wedge \eta \leq h\,|\,\xi(o)=x] = \\ &2Q \int_o^\infty \frac{P[\gamma_1 \wedge u \leq h\,|\,\xi(o)=x]}{h} \; g_2\, e^{-g_2 \cdot u}\, du \end{aligned}$$

This last equality uses the independence assumptions between M and W.
From 2.6

$$\frac{1}{h} P[\gamma_1 \wedge u \leq h | \xi_1(o) = x] \to 0$$

and we derive from Lebesgue's bounded convergence theorem that

$$\exists \lim_{h\to 0} \frac{1}{h} E[f(\xi(h))-f(\xi(\gamma_1 \wedge \eta \wedge h) | \xi(o) = (x,o)] = 0$$

Consider now the first term in the RHS of (2.7). From our independence assumptions, we see that it is equal to

$$\text{(2.8)} \qquad E\left[\left(\int_o^{\gamma_1 \wedge h} L_X f(\xi_1(s),0)ds\right) \cdot e^{-g_2(h\wedge\gamma_1)} | \xi_1(o)=x\right]$$
$$+ E\left[\int_o^{\gamma_1 \wedge h} g_2 \, e^{-g_2 u} \int_o^{\infty} (f(\xi_1(u),z)-f(x,o))dH_2(z)\right]$$

The first expression in (2.4) is obtained by dividing (2.8) by h, letting h go to zero and applying Lebesgue's bounded convergence theorem in each term. The remaining of the proof follows using similar arguments. □

Lemma 4

Until the assumption of theorem 3, $\exists K > 0$ such that $\forall t > 0$, $\forall x \quad \mathbb{R}^+ \times \mathbb{R}^+$

$$\frac{1}{t} |E[f(\xi(t))-f(\xi(o)) | \xi(o)=x] \leq K \qquad \square$$

Proof

Let $(D_n)_{n \geq 1}$ be the sequence of successive jumps of $(\xi(t))_{t \geq 0}$.
Denoting as $\alpha(t)$; $t \geq 0$, the counting measure of $(D_n)_{n \geq 1}$ – i.e. the number of points in the interval $[0,t]$ – we have :

$$\text{(2.9)} \qquad \alpha(t) \leq (\mu + \lambda)t$$

Note that there is as a finite number of points of $(D_n)_{n \geq 0}$ in the interval $[0,t]$. Hence, we may write :

$$(2.10)\qquad \begin{cases} f(\xi(t))-f(\xi(o)) = f(\xi(D_1 \wedge t^-))-f(\xi(o)) + S \\ \text{where} \\ S = \sum_{n=1}^{\infty} f(\xi(D_{n+1} \wedge t^-))-f(\xi(D_n \wedge t)) + \end{cases}$$

We have

$$(2.11)\qquad |E[S]| \le 2\,Q \sum_{n\ge 1} n \frac{((\lambda+\mu)t)^n}{n!} e^{-(\lambda+\mu)t} = 2Q\,(\lambda+\mu)t$$

We determine now an upperbound for the absolute value of

$$E[f(\xi(D_1 \wedge t^-)-f(\xi(o))\,|\,\xi(o)=x] \overset{\text{def}}{=} e$$

<u>For $x = (0,0)$</u>, then e is equal to zero.

<u>For $x \in$ "I"</u>, if $x=(y,0)$, $y > 0$, then e is equal to

$$E[f(\xi_1(\delta \wedge \gamma \wedge t^-),o)-f(\xi_1(0),0))\,|\,\xi_1(o) = y]$$

where - ξ_1 the solution of the linear S.D.E originated from y

- δ is an exponentially distributed random time (parameter g_2), independent of $(\xi(t))_{t\ge 0}$.
- γ is the hiting time of $\{0\}\times\{0\}$ by ξ.

Using Ito's formula we get :

$$|e| = |E\,[\int_o^{\delta\wedge\gamma\wedge t} L_X f(\xi_1(s),0)ds\,|\,\xi_1(o)=y]| \le \{\sup_{u\in\text{"I"}} |L_X f(u)|\}t$$

<u>For $x \in$ "II"</u>

If $x = (0,y)$ then

$$|e| = E\,[\int_o^{\delta\wedge\gamma\wedge t} L_Y f(0,\xi_2(s))ds\,|\,\xi_2(o)=y] \le \sup_{u\in\text{"2"}} |L_Y f(u)|\}t$$

For $x \in D$ then e is equal to (using the same notation as above)

$$E[f(\xi(\delta \wedge \gamma \wedge t^-))-f(\xi(\tau \wedge t))|\xi(o)=x]$$

$$+ E[f(\xi(\tau \wedge t))-f(\xi(o))|\xi(o)=x]$$

the second term satisfies :

$$|E[\int_o^{t \wedge \tau} Lf(\xi(s))ds]| \leq t.\{\sup_{x\in D} |Lf(x)|\}$$

and the first one

$$|E[f(\xi(\delta \wedge \gamma \wedge t^-)-f(\xi(\tau \wedge t))|\xi(o)=x]| \leq$$

$$t\{\sup_{u\in I} |L_X f(u)| + \sup_{u\in II} |L_Y f(u)|\}$$

completing the proof. □

Lemma 5. For f as in theorem 3, If there exists an invariant probability measure ν on $\mathbb{R}^+ \times \mathbb{R}^+$ for the Markov process $(\xi(t))_{t \geq 0}$ such that :

$$\iint_{\mathbb{R}^+ \times \mathbb{R}^+} \mathcal{A}f(x,y)\, d\nu(x,y) = 0 \qquad \square \tag{2.12}$$

Proof : Assume ν is an invariant measure, then $\forall\ t > 0$

$$\iint \frac{1}{t} E[f(\xi(t))-f(\xi(o))|\xi(o)=(x,y)]d\nu(x,y) = 0 \tag{2.13}$$

From theorem 3, we have, pointwise :

$$\lim_{t\to 0} \frac{1}{t} E[f(\xi(t))-f(\xi(o))|\xi(o)=x,y] = \mathcal{A}f(x,y)$$

Due to Lemma 4, Lebesgue's bounded convergence theorem applies when t converges to zero in (2.13) completing the proof. □

Theorem 6.

In the particular case where the coefficients B, Z, b_i, σ_i do not depend on x, If there exists an invariant measure ν for the Markov process $(\xi(t), t \geq 0)$ then the following Laplace Stieltjes transforms (defined for $Re(s) \geq 0$, $Re(t) \geq 0$) .

$$(2.14) \quad \begin{cases} I(s,t) = \iint_{\mathbb{R}^{+*} \times \mathbb{R}^{+*}} e^{-(sx+ty)} \, d\nu(x,y) \\ F(s) = \int_{\mathbb{R}^{+*}} e^{-sx} \, d\nu(x,\{o\}) \\ G(x) = \int_{\mathbb{R}^{+*}} e^{-ty} \, d\nu(\{o\},y) \\ J = \nu(\{o\},\{o\}) \end{cases}$$

satisfy the following functional equation

$$(2.15) \quad \begin{cases} I(0,0) + F(0) + G(0) + J = 1 \\ I(s,t)[\frac{1}{2}(as^2+2bst+ct^2)-ds-et] \\ + F(s)[\frac{1}{2} a_1 s^2 - b_1 s + g_2(h_2^*(t)-1)] \\ + G(t)[\frac{1}{2} a_2 t^2 - b_2 t + g_1(h_1^*(s)-1)] \\ + J\,[q_1(k_1^*(s)-1) + q_2(k_2^*(t)-1) = 0 \end{cases}$$

where $h_i^*(s)$ (resp. $k_i^*(s)$) denotes the Laplace Stieltjes transform of H_i (resp. K_i). □

Proof The complex valued function :

$$f(x,o) = e^{-(sx+ty)} \quad \begin{cases} x,y \geq 0 \\ Re(s),\ Re(t) \geq 0 \end{cases}$$

belongs to $\mathcal{D}(\mathcal{A})$ and we have (from theorem 3)

$$(2.16)\quad \begin{cases} x > 0,\ y > 0 & \mathcal{Q}f(x,y) = e^{-(sx+ty)}[\frac{1}{2}(as^2+2bst+ct^2)-ds-et] \\ y > 0 & \mathcal{Q}f(o,y) = e^{-sy}[\frac{1}{2}\ a_2t^2-d_2t+g_1(h_1^*(s)-1)] \\ x > 0 & \mathcal{Q}f(x,o) = e^{-tx}[\frac{1}{2}\ a_1s^2-d_1s+g_2(h_2^*(t)-1)] \\ & \mathcal{Q}f(o,o) = [q_1(k_1^*(s)-1)+q_2(k_2^*(t)-1)] \end{cases}$$

(2.15) is obtained from (2.12) and (2.16). □

It is easy to see that we introduce no restriction when assuming $b \leq 0$.

SECTION II

I. ANALYTICAL PRELIMINARIES

We consider in this section the functional equation determined in theorem 6

$$(1.1) \qquad I(s,t)\,R(s,t) = A(s,t)F(s) + B(s,t)G(t) + C(s,t)\,J = 0$$

The unknown functions $I(s,t)$, $F(s)$, $G(t)$ are assumed to be analytic in the domain $Re(s) \geq 0$, $Re(t) \geq 0$ and to satisfy $I(0,0)+F(0)+G(0)+J = 1$; The known functions are given by :

$$(1.2) \qquad \begin{cases} 2R(s,t) = as^2+2bst+ct^2-2ds-2et \\ 2A(s,t) = a_1s^2-2b_1s+2g_2(h_2^*(t)-1) \\ 2B(s,t) = a_2t^2-2b_2t+2g_1(h_1^*(s)-1) \\ C(s,t) = q_1(k_1^*(s)-1)+q_2(k_2^*(t)-1) \end{cases}$$

where the known (but unspecified) functions $h_1^*(s)$, $k_1^*(s)$, $h_2^*(t)$ and $k_2^*(t)$ are supposed to be analytic in the right half plane. To solve (1.1), we follow the principles given in [FA,IA]

I.1. $R(s,t) = 0$

We examine the algebraic curve defined by

$$(1.3) \qquad R(s,t) = 0$$

in the whole complex plane. Solving for t we obtain the following multi-valued algebraic function :

$$(1.4) \qquad T(s) = \frac{e - bs \pm \sqrt{(b^2-ac)s^2+2s(dc-be)+e^2}}{c}$$

From here on, for $z = \rho e^{i\theta}$, $\rho \geq 0$, $-\Pi < \theta \leq \Pi$, $\sqrt{z}$ will be taken as $\rho^{\frac{1}{2}} e^{i\frac{\theta}{2}}$ (so that $\mathrm{Re}(\sqrt{z}) \geq 0$, $\forall z \in \mathbb{C}$. The two branches give a two sheeted covering of the complex plane.

Lemma 7. The algebraic function $T(s)$ has two real branch points $s_2 < 0 < s_1$ □

Proof The branch points are the zeros s_1 and s_2 of the discriminant

$$\Delta(s) \overset{\mathrm{def}}{=} (b^2-ac)s^2 + 2s(dc-be) + e^2,$$

that is :

$$(1.5) \qquad \begin{cases} s_1 = \dfrac{be-cd-\sqrt{(cd-be)^2-e^2(b^2-ac)}}{b^2-ac} \\[2ex] s_1+s_2 = -2\,\dfrac{(dc-be)}{b^2-ac} \\[2ex] s_1 s_2 = \dfrac{e^2}{b^2-ac} \end{cases}$$

The positionsof these zeros are given by the sign of b^2-ac which is in turn determined by the ellipticity assumption on the two dimensional diffusion. □

Similar propostitions apply to $S(t)$. We denote as t_2 and t_1 the two branch points of $S(t)$ $(t_2 < 0 < t_1)$.

Lemma 8. The equation $R(s,t) = 0$ has one root $T_1(s)$ which is an analytic algebraic function of s in the whole complex plane cut along $]-\infty,s_2]$ and $[s_1,\infty]$ and satisfies :

$$(1.6) \qquad \begin{cases} \mathrm{Re}(T_1(s)) \geq 0 \\ \text{for } s = ix,\ x \in \mathbb{R} \end{cases}$$ □

Proof $T(ix) \overset{\mathrm{def}}{=} \alpha(x) + i\beta(x)$ verifies :

$$c(\alpha^2-\beta^2+2i\alpha\beta) + 2(bix-e)(\alpha+i\beta)-ax^2-2dix = 0$$

That is, separating the real and imaginary parts :

$$(1.7) \qquad c\alpha^2 - 2e\alpha - (ax^2+c\beta^2+2b\beta x) = 0$$

$$(1.8) \qquad \beta(e-c\alpha) = x(b\alpha-d)$$

The quantity $ax^2+c\beta^2+b\beta x$ being always positive (due to ellipticity), (1.7), taken as an equation in α, has already two real roots with opposite signs Q.E.D.

□

With our convention concerning the square root of a complex number, (1.6) yields :

$$(1.9) \qquad T_1(s) = \frac{e - bs + \sqrt{\Delta(s)}}{c}$$

Denoting as $T_2(s)$ the other root, the following inequality holds :

$$(1.10) \qquad \forall\, s \in \mathbb{C} \quad \mathrm{Re}(T_1(s)) \geq \mathrm{Re}(T_2(s))$$

<u>Lemma 9.</u> T_1 and T_2 map the cut $[s_1,\infty[$ (resp. $]-\infty,s_2]$) onto the right (resp. left) branch of the hyperbola

$$(1.11) \qquad \begin{cases} \left(x - \dfrac{ae-bd}{ac-b^2}\right)^2 = y^2.A^2 + K^2 \quad \text{where} \\ A^2 = \dfrac{b^2}{ac-b^2} > 0 \\ K^2 = \dfrac{d^2}{c(b^2-ac)^2} \cdot (ae^2-2bde+cd^2) > 0 \end{cases}$$

This hyperbola will be denoted as τ. □

<u>Proof</u> Take s on the upper part of the cut $[s_1,\infty[$ $T_1(s)$ and $T_2(s)$ are complex conjugate. Denote as x and y the respective real and imaginary parts of $T_1(s)$. From the equations :

$$T_1(s) + T_2(s) = 2\,\frac{e-bs}{c} = 2x$$

$$T_1(s).T_2(s) = \frac{as^2-2ds}{c} = x^2 + y^2$$

We see that x and y are real solutions of

$$[c(b^2-ac)]x^2 + (b^2c)y^2 + 2c(ae-bd)x = e(ae-2bd)$$

(1.11) readily follows.

We shall denote as σ the hyperbola $\{S(t),\ t \in]-\infty,t_2] \cup [t_1,\infty[\}$

Lemma 10. Let D_σ (resp. G_σ) be the infinite domain inside the right (resp. left) branch of the hyperbola σ, we have :

$$S_1T_1(s)\begin{cases} = \quad s \quad \text{for} \quad s \in D_\sigma \\ = \quad -s + \frac{2d}{a} - \frac{2b}{a}\, T_1(s) \quad \text{for } s \in \bar{D}_\sigma \quad (=\mathbb{C} - D_\sigma) \end{cases}$$

$$S_2T_1(s)\begin{cases} = \quad s \quad \text{for} \quad s \in \bar{D}_\sigma \\ = \quad -s + \frac{2d}{a} - \frac{2b}{a}\, T_1(s) \quad \text{for } s \in D_\sigma \end{cases}$$

$$S_1T_2(s)\begin{cases} = \quad s \quad \text{for} \quad S \in \bar{G}_\sigma \\ = \quad -s + \frac{2d}{a} - \frac{2b}{a}\, T_2(s) \quad \text{for } s \in G_\sigma \end{cases}$$

$$S_2T_2(s)\begin{cases} = \quad s \quad \text{for} \quad S \in G_\sigma \\ = \quad -s + \frac{2d}{a} - \frac{2b}{a}\, T_2(s) \quad \text{for } s \in \bar{G}_\sigma \end{cases}$$

Proof : We first determine the value of S_1T_1 and S_2T_1 in D_σ. Let Id denote the Identity function. We calculate the value of S_1T_1-Id for $s \in [s_1,\infty[$. T_1 maps $[s_1,\infty[$ onto the right branch of τ. For a given t belonging to this branch, we have

$$S_1(t)\ .\ S_2(t) = \frac{ct^2-2et}{a}$$

$$S_1(t) + S_2(t) = 2\,\frac{d-bt}{a}$$

Hence, for $\hat{t} = \frac{e-bs_1}{c} = T_1(s_1)$, it holds

$$S_1(\hat{t}) + S_2(\hat{t}) = 2\,\frac{d - b.\frac{e-bs_1}{c}}{a}$$

Furthermore, one of the two complex numbers $S_1(\hat{t})$ and $S_2(\hat{t})$ coïncides with s_1 (since $R(s_1,\hat{t}) = 0$) and hence, the other one is

$$u = -s_1 + 2\,\frac{d - b\,\frac{e-bs_1}{c}}{a}$$

since this last real number is smaller than s_1 (this is obtained from equation (1.5) and from the equivalence $u < s_1 \iff s_1 > \frac{dc-eb}{ac-b^2}$) and since $S_1(\hat{t}) \geq S_2(\hat{t})$ (see (1.10)), we necessarily have :

$$S_1T_1(s_1) = s_1 \geq S_2T_1(s_1) = u$$

So that $S_1T_1-Id = 0$ on the cut $[s_1,\infty[$.

Consider now the following point in the s plane : $\hat{s} = \frac{d-bt_1}{a}$. Using similar arguments, we show that $T_1(\hat{s}) = t_1$. Accordingly $S_1T_1(\hat{s}) = \hat{s}$ so that $S_1T_1-Id = 0$ on the right branch of σ. Hence $S_1T_1-Id = 0$ on the boundary of the infinite domain $D_\sigma - [s_1,\infty[$. Since this function is analytic in this domain and continuous on the boundaries, it reaches its maximum on the boundary (maximum modulus principle). We deduce from the preceding results that $S_1T_1-Id = 0$ inside D_σ. Similarly for s on the boundary of $D_\sigma - [s_1,\infty[$, we have :

$$S_2T_1(s) + s - 2\,\frac{d-bT_1(s)}{a} = 0$$

This equality is also extended to $s \in D_\sigma$.

To determine now S_1T_1 and S_2T_1 on $\bar{D}_\sigma$, we first establish the following inequality :

$$S_1T_1(s_2) = - s_2 + 2\,\frac{d - b\,\frac{e-bs_2}{c}}{a} > s_2 = S_2T_1(s_2)$$

Then notice that S_1T_1 is an analytic function inside the domain $\bar{D}_\sigma -]-\infty,s_2]$, continuous on the cut (but not on the right branch of σ).

An analytic continuation argument yields

$$S_1T_1(s) = -s + 2\,\frac{d-bT_1(s)}{a} \qquad \forall\ s \in \mathrm{Int}(\bar{D}_\sigma)$$

The determination of S_2T_1 is similar . □

I.2. Relative positions of the curves

I.2.1. The hyperbola and the cuts

Lemma 11. The cut $[t_1,\infty[$ (resp $]-\infty,t_2]$) is situated inside the right branch (left branch) of the hyperbola τ. □

Proof The intersection of the right branch of the hyperbola τ with the real axis is situated at $\frac{e-bs_1}{c}$. Using (1.5)

$$\frac{e-bs_1}{c} = \frac{bd - ae - \sqrt{\frac{b^2}{ac}}\ \sqrt{(ae-bd)^2-d^2(b^2-ac)}}{b^2 - ac}$$

The right hand side of this expression is smaller than t_1 due to ellipticity. □

I.2.2. The transforms at the imaginary axis

We just give obvious results relative to the s plane :

- These transforms are symmetrical relative to the real axes.

- $T_1(o) = \max\ (\frac{2e}{c},o))$
  $(T_2(o) = \min\ (\frac{2e}{c},o))$

- $(T_1(x),\ x \in \mathbb{R})$ intersects the real axis (excluding $T_1(o)$) at the point $\alpha = \frac{b}{d}$ only if $d < 0$ and $e < 0$ or $d < 0$, $e > 0$ and $dc-2eb < 0$. Otherwise there is no other intersection than $T_1(o)$.

- Asymptote : $\dfrac{-b}{\sqrt{ac-b^2}}$ for $x \to +\infty$ and $\dfrac{b}{\sqrt{ac-b^2}}$ for $x \to -\infty$

I.2.3. The hyperbola and T_i ($]s_2, s_1[$)

Lemma 12. The derivative of $T_i(s)$ does not vanish on the open real interval $[s_2, s_1[$, but for $s = \frac{d-bt_i}{a}$. We have the following table :

	s_2		$\frac{d-bt_2}{a}$		$\frac{d-bt_1}{a}$		s_1
$T_1'(s)$		+		+	0	-	
$T_1(s)$		↗		↗	t_1	↘	
$T_2'(s)$		-		+		+	
$T_2(s)$		↘	t_2	↗		↗	

□

Proof :

$$T'(s) = \frac{-b\sqrt{\Delta(s)} \pm \left[s(b^2-ac)+(dc-be)\right]}{c\sqrt{\Delta(s)}}$$

If $\hat{s}$ / $T'(\hat{s}) = 0$, then $\hat{s}$ $= \dfrac{dc - be \pm \sqrt{\frac{b^2}{ac}}\ \sqrt{(dc-be)^2 + e^2(ac-b^2)}}{b^2 - ac}$

$$= \left\{\frac{d-bt_1}{a}, \frac{d-bt_2}{a}\right\}$$

Since $\mathrm{Re}(\sqrt{\Delta(\hat{s})}) \geq 0$, we necessarily have $T_i(\hat{s}) = \dfrac{d-bt_i}{a}$

□

I.2.4. The hyperbolas and the imaginary axis

Lemma 13.

If $e < 0,\ d < 0$ then $\frac{e-bs_1}{c} < 0$

If $e > 0,\ d < 0$ then $\frac{e-bs_1}{c} > 0$

If $e < 0,\ d > 0$ and $ae - 2bd < 0$ then $\frac{e-bs_1}{c} < 0$

If $e < 0,\ d > 0$ and $ae - 2bd > 0$ then $\frac{e-bs_1}{c} > 0$

Proof $\Delta(\frac{e}{b}) = -\frac{ec}{b^2}[ae - 2bd]$ □

Similarly, we have :

If $e < 0,\ d < 0$ $\frac{d-bt_1}{a} < 0$

If $e > 0,\ d < 0$ and $cd - 2eb, < 0 = \frac{d-bt_1}{a} < 0$

If $e > 0,\ d < 0$ and $cd - 2eb, < 0 = \frac{d-bt_1}{a} > 0$

If $e < 0$ et $d > 0$ $\frac{d-bt_1}{a} > 0$.

Lemma 14.

(We denote as σ^+ (resp. σ^-) the right (resp. left) branch of σ.)

If $(2b^2-ac) > 0$ (resp. $(2b^2-ac) < 0$) then $T_2(\sigma^+)$ is on the left (resp. the right) of $T_1(\sigma^-)$. If $2b^2=ac$, then the curves

Proof : We have :

$$\left\{\begin{array}{l} \dfrac{d-bt_1}{c} = \dfrac{cd - be + \sqrt{\dfrac{b^2}{a}\Delta}}{ac - b^2} \\ \text{where} \quad \Delta = ae^2 - 2bde + cd^2 > 0 \end{array}\right.$$

From (1.4) :

$$T\left(\frac{d-bt_1}{c}\right) = \frac{ae - bd}{ac - b^2} - \frac{b^2 \pm (b^2-ac)}{c} \cdot \sqrt{\frac{\Delta}{a}}$$

From lemma 10,

$$T_2\left(\frac{d-bt_1}{c}\right) = \frac{ae - bd}{ac - b^2} - \frac{2b^2 - ac}{c} \cdot \sqrt{\frac{\Delta}{a}}$$

and

$$T_1\left(\frac{d-bt_1}{c}\right) = \frac{ae - bd}{ac - b^2} + \frac{2b^2 - ac}{c} \cdot \sqrt{\frac{\Delta}{a}}$$

completing the proof. □

II. ANALYTIC CONTINUATION OF G(t) and F(s)

From now on, it will be assumed that the functions $h_1^*(s)$, $k_1^*(s)$, $h_2^*(t)$, $k_2^*(t)$ are meromorphic in the whole complex plane. In this case :

Theorem 15 : Assume that (1.1) has a solution. Then the function F(s) resp. G(t)) which is analytic for $Re(s) \geq 0$ (resp. $Re(t) \geq 0$) can be continued as a meromorphic function to the whole complex plane cut along $]-\infty,s_2]$ (resp. $]-\infty,t_2]$).

Proof : For a given s such that $Re(s) \geq 0$ and $Re(T_i(s)) \geq 0$ simultaneously, (1.1) implies that

(2.1)
$$\begin{cases} F(s) + \beta(s,T_i(s))\, G(T_i(s)) + \gamma(s,T_i(s)) = 0 \\ \text{where} \\ \left.\begin{array}{l} \beta(s,t) \overset{\text{def}}{=} \dfrac{B(s,t)}{A(s,t)} \\ \gamma(s,t) \overset{\text{def}}{=} \dfrac{C(s,t)}{A(s,t)} \end{array}\right\} \text{ are meromorphic functions} \end{cases}$$

In the first step we show that $G(T_1(s))$ can be continued as a meromorphic function for $s \in D_\sigma$.

T_1 maps conformaly D_σ onto D_τ (see lemmas 10 and 11). We consider the following cases

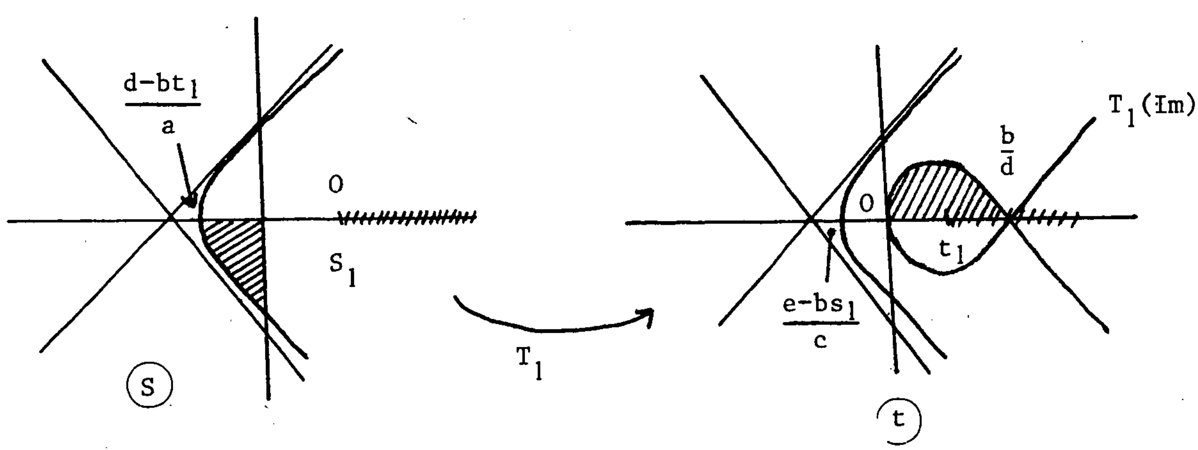
*) $\frac{e-bs_1}{c} \geq 0$. Then $\forall\ s \in D_\sigma$, $\mathrm{Re}(T_1(s)) \geq 0$ so that $G(T_1(s))$ is well defined in D_σ.

*) $\frac{e-bs_1}{c} < 0$ and $\frac{d-bt_1}{a} \geq 0$. Then $\forall\ s \in D$, $\mathrm{Re}(s) \geq 0$ so that $F(s)$ is well defined. Equation (2.1) defines directly the analytic continuation of $G(t)$ to $T_1(D_\sigma)$

*) $\frac{e-bs_1}{c} < 0$ and $\frac{d-bt_1}{a} < 0$ (This may only be the case if $e < 0$ and $d < 0$). Let

$$\mathcal{J} = \{s \ / \ \mathrm{Re}(s) < 0 \quad \text{and} \quad s \in D_\sigma\}$$

Using (2.1), we continue $G(T_1(s))$ as a meromorphic function to $D_\sigma - \mathcal{J}$. For $s \in \mathcal{J}$, notice that $\mathrm{Re}(T_1(s)) \geq 0$, so that $G(T_1(s))$ is continued to D . Hence $G(t)$ is continued as a meromorphic function to $\mathcal{D}_0 \stackrel{\text{def}}{=} D_\tau$.



<u>In the second step, $G(t)$ is continued as a meromorphic function to the domain $\mathcal{D}_1$ bounded by</u> $T_2(\sigma^+)$ <u>and τ^+</u>. This is obtained from (2.1) :

$$F(s) + \beta(s,T_2(s))\ G(T_2(s)) + \gamma(s,T_2(s)) = 0$$

and from the fact that F is continued as a meromorphic function to D_σ .

<u>In the last step $G(t)$ is continued to the domain $\mathcal{D}_2$ bounded by</u> $]-\infty,t_2]$ <u>and</u> $T_2(\sigma^+)$. Let $A = \{t/t$ is located between $T_2(\sigma\ )$ and τ , and $S_2(t) \in \bar{G}_\sigma\}$.

Assume first that A is empty. This means that $S_2(\tau^+)$ is on the left of τ^- (see lemma 14). In this case, F is already continued to the domain $\bar{G}_\sigma$ by (the symetrical of) step 2. Using the monotonicity of T_2 in $\bar{G}_\sigma$ we directly continue G as a meromorphic function to $\mathbb{C}$ cut along $]-\infty,t_2]$ from (2.1).

Assume now that A is not empty. Then, $\forall t \in A$, $\exists\, s = S_2(t)$ such that $s \in \bar{G}_\sigma$ and $T_1(s)$ $(=t)$ is between $T_2(\sigma^+)$ and τ^+ (this is a consequence of lemma 10).

Consider now the following equation obtained from (2.1)

$$\beta(s,T_1(s))\ G(T_1(s)) + \gamma(s,T_1(s)) = \beta(s,T_2(s))\ G(T_2(s)) + \gamma(s,T_2(s)) \tag{2.2}$$

$\forall\ t \in A$, we get from (2.2)

$$G(T_2S_2(t)) = G(t)\ \frac{\beta(S_2(t),t)}{\beta(S_2(t),T_2S_2(t))} + \frac{\gamma(S_2(t),t) - \gamma(S_2(t),T_2S_2(t))}{\beta(S_2(t),T_2S_2(t))} \tag{2.3}$$

This last relationship is first continued to the domain $B = \{t/t$ is between $T_2(\sigma^+)$ and the cut $[t_1,\infty[$ and $t \in \bar{G}_\sigma\}$.
Since $T_2\ S_2$ maps domain B onto the domain $\mathcal{D}_2$ between the cut $]-\infty,t_2]$ and $T_2(\sigma^+)$, (2.3) definies the meromorphic continuation of G onto $\mathcal{D}_2$.

III. DETERMINATION OF F(s) : REDUCTION TO A NON HOMOGENEOUS RIEMANN-HILBERT BOUNDARY VALUE PROBLEM ON THE RIGHT-HAND BRANCH OF HYPERBOLA σ.

We proceed again as in [FA,IA], $G(t)$ must be analytic for the $Re(t) > 0$: in particular, $G(t)$ is continuous on the cut $[t_1,+\infty[$, which yields

$$G^+(t) = G^-(t), \quad t \in [t_1,\infty[\tag{3.1}$$

where $G^+(t)$ [resp. $G^-(t)$] denotes the limit of $G(t)$ from above [resp. from below] the cut.

Using the symetric of (2.1), (3.1) and the functions $S_1(t)$ and $S_2(t)$ (which are conjugate for $t \in [t_1,\infty[$) defined in section 1, we get for $F(s)$ the following boundary value condition on σ^+, in the s-plane :

$$U(s)F(s) - V(s) = U(\bar{s})\, F(\bar{s}) - V(\bar{s}), \quad s \in \sigma^+ \tag{3.2}$$

where $\bar{s}$ is the conjugate value of s and

$$\begin{cases} U(s) \overset{\text{def}}{=} \dfrac{A(s,T_1(s))}{B(s,T_1(s))} \\ V(s) \overset{\text{def}}{=} -\dfrac{C(s,T_1(s))}{B(s,T_1(s))} \end{cases}$$

Moreover, the coefficients of the known functions $U(s)$ and $V(s)$ are real and so is $G(t)$, $t \in [t_1,\infty[$. This allows us to rewrite (3.2) as

$$\mathrm{Im}\,[U(s).F(s)] = \mathrm{Im}\,[V(s)], \quad s \in \sigma^+ . \tag{3.3}$$

$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.

Our problem has almost been reduced to what is known as Hilbert's or Riemann-Hilbert's problem P : P ≡ "Find a function $f(z)$, analytic inside a simple closed smooth curve $\mathscr{L}$ and satisfying on $\mathscr{L}$ a boundary condition of the form

(3.4) $\quad \mathrm{Re}[\varphi(z).f(z)] = \Psi(z), \quad z \in \mathcal{L}$, where $\varphi(z)$ and $\Psi(z)$

are known functions defined a priori only on $\mathcal{L}$ and regular enough [generally they satisfy Hölder's condition, but it is possible to relax this restriction retaining only the natural condition of continuity]. Re(z) is the real part of z. The number of independent solutions of 3.4 depends on what is called the index ξ of the problem

(3.5) $\quad \xi = \frac{-1}{\pi} \arg [\varphi(z)]_{\mathcal{L}}$, where $\arg [\varphi(z)]_{\mathcal{L}}$ is the variations of the argument of $\varphi(z)$ when z moves along the curve $\mathcal{L}$ in the positive (counterclockwise) direction. [See Musk., Gak...]
[To have a well defined ξ, it is necessary to get rid of the zeros and the pôles of $\varphi(z)$].

- There are the following general results :

i) for $\xi \geq 0$, the homogeneous problem P (i.e $\Psi(z) \equiv 0$) has exactly $\xi+1$ linearly independent solutions. The general solution is given by $\theta(z) = f(z).(c_o z^{\xi} + c_1 z^{\xi-1} \ldots + c_{\xi})$ where the c_k, $k=1,\ldots\xi$ are constants subject to $c_k = \bar{c}_{\xi-k}$ but otherwise arbitrary.

ii) for $\xi \leq -2$, the homogeneous problem P has no solutions different from zero.

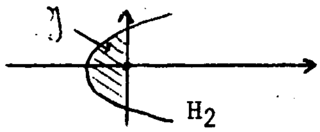
iii) For $\xi \geq 0$, the non homogeneous problem P has a always a solution. The general solution involves $(\xi+1)$ real constants.

iv) For $\xi \leq -2$, problem P has a solution if and only if $-\xi-1$ "orthogonality" conditions are satisfied by the free term $\Psi(z)$.

When the contour $\mathcal{L}$ is not closed, it can be convenient to reduce the problem to that considered for a closed contour : we complete the contour by an arbitrary line so that one closed curve is constructed.

One can show that the solution of the Riemann-Hilbert problem for an open contour is independant of the form of the curve supplementing the considered contour to a closed one [Gak.]

Let us assume now that σ^+ and the imaginary axis intersect each other and denote by $\mathfrak{J}$ the region $D_\sigma \cap \{s \mid \mathrm{Re}(s) < 0\}$



(see the preceding sections for the conditions of having a non empty $\mathfrak{J}$).
One easily verifies that condition (#) holds when $|s| \to \infty$, $|\arg s| < \frac{\pi}{2}$.

To get a problem of type P, we proceed now in two steps :

A. Map conformally the domain inside D_σ onto the inside of the unit circle $|w| \leq 1$

B. Get rid of the eventual pôles de F(s) in $\mathfrak{J}$.

A. The conformal mapping

For sake of clarity, we consider the domain D inside the right branch H of the reduced hyperbola in the s-plane, i.e.

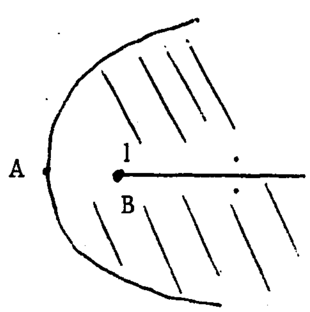
$$\frac{X^2}{\cos^2 \alpha} - \frac{Y^2}{\sin^2 \alpha} \geq 1 , \quad 0 < \alpha < \frac{\pi}{2} \quad , \quad X > 0 .$$

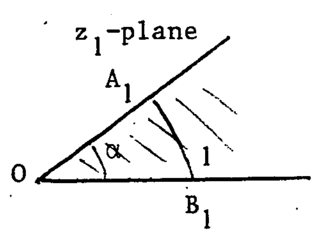
The following arguments are well-known in the theory of conformal mappings :

1) Using Joukowski transform

$z_1 = s + \sqrt{s^2-1}$ maps the upper half of D onto a domain $|z_1| > 1$, $0 < \arg z_1 < \alpha$, where $i \to i(1+\sqrt{2})$

2) Schwarz's symetry principle :

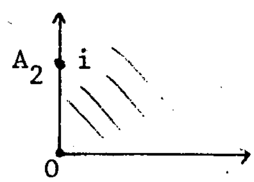


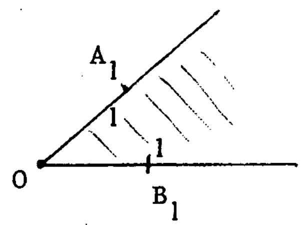


The domain which is symetrical to AB in the s plane is mapped onto a domain obtained by inversion w.r.t the arc A_1B_1.

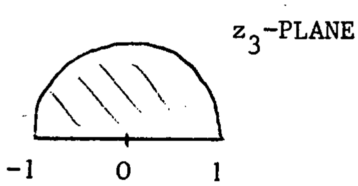
Hence we have a conformal mapping of D cut along $[1,+\infty[$ onto the sector $0 < \arg z_1 < \alpha$. The ray $\arg z_1 = \alpha$ corresponds to the hyperbola H - and the ray $\arg z_1 = 0$ corresponds to the cut $[1,+\infty[$, OB_1 [resp. $(B_1,+\infty)$] being the image of the lower [resp. upper] edge.

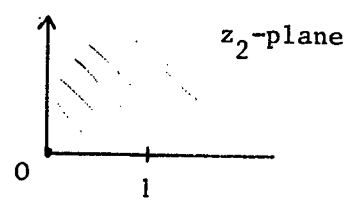
3) $z_2 = z_1^{\frac{\pi}{2\alpha}}$ maps the sector onto the first quadrant





4) $z_3 = \dfrac{z_2-1}{z_2+1}$ maps the first quadrant in the z_2-plane onto the upper-half of the unit circle.





The two edges of the cut $[1,\infty[$ in the original s-plane are then mapped onto the segment $[-1,+1]$.

5) $w=(z_3)^2$ "gathers" $[-1,0]$ and $[0,1]$ Hence giving a one to one transform of $[1,\infty[$ in the s-plane onto $[0,1]$ in the w-plane. The result follows

(3.6) $$w(s) = \left(\frac{z_1^{\frac{\pi}{2\alpha}} - 1}{z_1^{\frac{\pi}{2\alpha}} - 1} \right)^2$$, where $z_1 = s + \sqrt{s^2-1}$, the branch being chosen as above $[i \to i(1+\sqrt{2})]$.

s(w) will denote the inverse function.

The point at infinity on H, in the s-plane is mapped onto the point 1 on the real axis in the w-plane.

B. Introduce the new functions

$$(3.7)\qquad \begin{cases} f(\omega) = F(s(\omega)) \\ \Psi(\omega) = -\operatorname{Im} V(s(\omega)) \\ \varphi(\omega) = -\, i\, U(s(\omega)) \end{cases}$$

(3.3) gives

$$\operatorname{Re}[\varphi(\omega).f(\omega)] = \Psi(\omega), \quad \omega \in \mathcal{C}\,[\text{the unit circle}].$$

We must now ensure that the function sought is analytic in D.

Lemma 16

$$f(\omega) = \frac{f_o(\omega)}{\prod_{i=1}^{m} (\omega-\omega_i)} \quad , \text{ where}$$

- a) $f_o(\omega)$ is analytic inside $\mathcal{C}$.

- b) ω_i, $i=1,\ldots,m$ are obtained by conformal mapping from the points s_i in the s-plane which are :

  i) some of the eventual zeros of $A(s, T_1(s))$ in $\mathcal{J}$ for which $B(s,T_1(s))$ and $C(s,T_1(s))$ do not simultaneously vanish.

  ii) the eventual poles of $h_1^*(s)$ in $\mathcal{J}$.

Proof a) is obvious because F(s) tends to zero when $s \to \infty$ in D.

b) from the assumptions at the beginning of this section, the poles of $h_1^*(s)$ are real in $\mathcal{J}$: hence, $h_1^*(s) \geq 0$ for real s in $\mathcal{J}$ and $2g_1\, G(T_1(s)) + \pi_1$ does not vanish. The ω_i are not on the contour, but for some pathological cases excluded here. □

Setting

$$(3.8) \qquad \varphi_o(\omega) \overset{\text{def}}{=} \frac{\varphi(\omega)}{\prod_{i=1}^{m} (\omega-\omega_i)} ,$$

relation (3.3) yields

$$(3.9) \qquad \mathrm{Re}[\varphi_o(\omega).f_o(\omega)] = \Psi(\omega), \qquad \omega \in \mathcal{C}$$ which is the

typical problem P. □.

IV. FORMULAE FOR THE SOLUTION OF THE RIEMANN-HILBERT PROBLEM

Lemma 17. The index of the problem is given by :

$$\xi = \frac{1}{2\pi} \arg \left[\frac{\overline{\varphi_o(\omega)}}{\varphi_o(\omega)} \right]_{\mathcal{C}} \qquad \square$$

Proof : When $\omega \in \mathcal{C}$, the ratio $\frac{\overline{\varphi_o(\omega)}}{\varphi_o(\omega)}$ does not vanish : indeed ,

$$\frac{\overline{\varphi_o(\omega)}}{\varphi_o(\omega)} = \frac{\overline{U(s)}}{U(s)} \cdot \prod_{i=1}^{m} \frac{s-s_i}{\overline{s-s_i}} , \qquad \omega \in \mathcal{C} \text{ and } s = s(\omega) \in H \qquad \square$$

From the irreducibility of the original Markov process, we can assert that the index ξ is will never be positive : otherwise, there would be several "acceptable" invariant measures corresponding to the various ergodic classes. Hence $\xi \leq 0$ and the problem has at most one solution which, when it exists, is given by

$$(4.1) \qquad \boxed{f_o(\omega) = \frac{\phi(\omega)}{i\pi}\left[\rho(\omega) - 1_{(\xi=0)}\left(\frac{\rho(o)}{2} + D\right)\right]} \qquad , |\omega| < 1$$

where *) D is a real constant ;

$$*) \ 1_A = \begin{cases} 1, & \text{if the event A occurs ;} \\ 0 & \text{otherwise} \end{cases}$$

*) $\rho(\omega) = \int_{\mathcal{C}} \frac{\Psi(t)dt}{\varphi_o(t).\phi^+(t)\ (t-\omega)}$;

*) $\phi(\omega) = e^{\gamma(\omega)}$;

*) $\phi^+(t)$ is the boundary value of $\phi(\omega)$ when $\omega \to t$ from the inside of $\mathcal{C}$;

*) $\gamma(\omega) = \frac{1}{2\pi} \int_{\mathcal{C}} \theta(u) \frac{u+\omega}{u-\omega} du$;

*) $\theta(u) = \arg\left[-u^{-\xi} \frac{\overline{\varphi_o(u)}}{\varphi_o(u)}\right] = -i \log\left[-u^{-\xi}. \frac{\overline{\varphi_o(u)}}{\varphi_o(u)}\right]$, $|u| = 1$

Remarks

1) When $\xi = 0$, there always is a solution. The unknown constant D (coming in 4.1) is determined by the additional condition

$$\lim_{\substack{s\to\infty \\ Re(s)>0}} F(s) = \lim_{\omega\to 1} f(\omega) = 0$$

2) When $\xi < 0$ (ξ is obviously an even number on the contour, so that $\xi \leq -2$)

$-\xi - 1$ additional conditions must be satisfied by $\Psi(\omega)$ [Gak. Musk] to ensure the existence of an analytic function $f_o(\omega)$ in $\mathcal{C}$. These conditions have the form

$$\int_{\mathcal{C}} \frac{t^k\ (t)dt}{\varphi_o(t)\ \phi^+(t)} = 0, \quad k=0,1,\ldots-\xi-2 \tag{4.2}$$

3) One can simplify the computation of ξ as follows :

$$\xi = + \frac{1}{2\pi} \arg\left[\frac{U(S_1(t))}{\overline{U(S_1(t))}}\right]_{[-i\infty,+i\infty]} + \theta + 2[m + N_p - N_z] , \tag{4.3}$$

where : i) N_p[resp. N_z] is the number of poles [resp. zeros] in the right half t-plane cut along $[t_1,\infty[$.

ii) $(i\infty,-i\infty)$ is the imaginary axis described in the top-down direction.
The proof is a direct consequence of the "principle of the argument" in the right half plane cut along $[t_1,\infty[$.

iii) θ is the angle between the asymptote of hyperbola in the t-plane.

4) A few words about the ergodicity conditions of the studied diffusion process are now in order.
In a similar way, let η denote the index corresponding to the function $G(t)$.
The existence of $F(s)$ [resp. $G(t)$], analytic in the right half s-plane [resp. t-plane] , is equivalent to give the index ξ [resp. η] the "correct" value.

Computation of these indexes depends on 14 parameters in the general case (among which 4 given but unspecified functions) and is somehow numbersome.

We only emit one conjecture

- the ergodicity conditions are given by

$$
(4.4)\quad \begin{cases} \left.\dfrac{dA(s,T_1(s))}{ds}\right|_{s=0} < 0 \iff b_1 + g_2\, h_2^{*\prime}(o)\, \dfrac{bT_1(o)-d}{e-cT_1(o)} < 0 \\[2ex] \left.\dfrac{dB(s_1(t),t)}{dt}\right|_{t=0} < 0 \iff b_2 + g_1\, h_1^{*\prime}(o)\, \dfrac{bS_1(o)-e}{d-cS_1(o)} < 0 \end{cases}
$$

where (recall) $S_1(o) = \max\left(0,\frac{2d}{c}\right)$ and $T_1(o) = \max\left(0,\frac{2e}{c}\right)$.

This conjecture [FA,IA] has already been proved for several functional equations analoguous to (1.1) - section II, which arise in queueing theory, where the range of the variables (s,t) is most often the unit circle instead of the right half plane.

5) To get G(t), one simply writes from (1.1) - section II

$$B(s,t)G(t) = -F(s)A(s,t) - \gamma\, C(s,t), \text{ where } s = S_1(t).$$

The constant γ is obtained from the normalizing equation

$$I(o,o) + F(o) + G(o) + \gamma = 1.$$

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